THE STRUCTURE OF SHOCK WAVES

(O STRUKTURE UDARNYKH VOLN)

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In this paper we discuss systems of equations of the following type

$$\frac{\partial}{\partial t} A'_i + \frac{\partial}{\partial x} B'_i = \frac{\partial}{\partial x} L_{ij} \frac{\partial u_j}{\partial x} \qquad \left(A'_i = \frac{\partial A}{\partial u_i},\ B'_i = \frac{\partial B}{\partial u_i}\right) \qquad (i, j = 1, \ldots, n) \qquad (1)$$

where A, B, and L_{ij} are given functions of the variables u_k, and the matrices $\| A''_{ij} \|$ and $\| L_{ij} \|$ are positive definite, i.e. for any values of z_i not simultaneously zero the following inequalities are fulfilled

$$A''_{ij} z_i z_j > 0, \qquad L_{ij} z_i z_j > 0$$

(In the present paper repeated indices denote summation). Godunov has demonstrated how the equations of one-dimensional gas dynamics [1], the equations of magnetohydrodynamics and several other systems of interest can be reduced to this form.

The matrix of the dissipation coefficients L_{ij} is not assumed symmetrical in this paper. The asymetry of the matrix L_{ij} may be due to the magnetic field. It is easy to see that under these conditions, system (1) is evolutionary in the sense of [2]. It is possible to show, moreover, that the system is dissipative in the sense of [3], i.e. any solution of the linearized system of the form $e^{i(kx - \omega t)}$, k real, (and therefore any solution which may be expanded as a Fourier integral) tends to zero when $t \to \infty$.

The following theorem is valid for systems of type (1).

Theorem. Suppose $a_\alpha(u_k)$ is one of the velocities of propagation of small disturbances, and it is a simple root of the characteristic equation

$$| B''_{ij} - a A''_{ij} | = 0 \qquad (2)$$

and let $da_\alpha \neq 0$ in the corresponding simple wave, i.e.

$$da_\alpha = (\partial a_\alpha / \partial u_j)\, du_j \neq 0$$

for du_j determined by the systems of equations

$$(B''_{ij} - a_\alpha A''_{ij})\, du_j = 0 \tag{3}$$

Then, if U is close to $a_\alpha(u_k^+)$, whilst $U > a_\alpha(u_k^+)$, where u_k^+ is some selection of variables, then for system (1) there exists a solution of the type $u_k = u_k(x - Ut)$, taking on values of u_k^+ at $x = \infty$, whilst when $x = -\infty$ the values of u_k^- are close to u_k^+. The values u_k^- satisfy the system of relations

$$B'_i(u_k^-) - UA'_i(u_k^-) = B'_i(u_k^+) - UA'_i(u_k^+) \equiv C_i \quad (i = 1, \ldots n) \tag{4}$$

which in the neighborhood of values of u_k^+ determine uniquely the values of $u_k^- \neq u_k^+$.

If all the $L_{ij} \to 0$ but the cosine of the angle between the vectors $\{L_{ij} z_j\}$ and $\{z_i\}$ does not tend to zero at any values of z_i, the continuous solution indicated above tends to a discontinuous one

$$u_k(x - Ut) = u_k^+ \quad \text{for } x - Ut > \lambda$$
$$u_k(x - Ut) = u^- \quad \text{for } x - Ut < \lambda$$

If we do not insist on demanding finiteness of the cosine of the angle between the vectors $\{L_{ij} z_j\}$ and $\{z_i\}$ we can make L_{ij} tend to zero in such a manner that the solution will not reduce to the above discontinuous solution and the width of the region within which at least one u_i differs from u_k^+ and u_k^- by a greater amount than some fixed value δ, may actually tend to infinity.

This theorem is close in its content to a theorem proved by Liubarskii [4] who studied the more general systems of equations of the hyperbolic type. However, when examining a system of equations of type (1) for the existence of the above described continuous solutions it is essential to fulfill a smaller number of supplementary conditions imposed on the system and on the given solution than in the case discussed by Liubarskii. In particular we do not require that the dispersion equation of the linearized system $D(i\omega, ik) = D(\nu U, -\nu) = 0$, in which the variables have been changed to $U = -\omega/k$, $\nu = -ik$ have only real roots ν for a given value of U. This condition may be unfulfilled for cases where there is a fairly high degree of asymetry of the matrices of the dissipative coefficients L_{ij}. Thus, for instance, according to [5] in which the structure of a magnetohydrodynamic shock wave was studied, when the mechanism of dissipation is given by the generalized Ohm's law this condition is not satisfied in a rarefied plasma situated in a strong magnetic field.

The proof of the above theorems is based on an extension of [6,7,8].

We now deal with the continuous solutions of system (1) which depend on $\xi = Ut - x$. These solutions satisfy a system of ordinary differential equations [1,6]

$$L_{ij} \frac{du_j}{d\xi} = P'_i, \qquad P \equiv UA - B + C_j u_j \tag{5}$$

which are obtained from Equation (1) by integrating with respect to ξ. On each integral curve of system (5) the inequality $dP/d\xi \geqslant 0$ is fulfilled. Actually [1,6]

$$\frac{dP}{d\xi} = P'_j \frac{du_j}{d\xi} = L_{ij} \frac{du_i}{d\xi} \frac{du_j}{d\xi} \geqslant 0 \tag{6}$$

whilst the equality will only hold when all the derivatives $du_k/d\xi$ are zero, i.e. at singular points of system (5). The code coordinates of the singular points of system (5) satisfy the equations $P'_i = 0$, which are identical with Equations (4).

The solution which reduces to u_k^+ when $x = \infty$ and to u_k^- when $x = -\infty$ is represented in the space u_k by an integral curve of system (5) which connects singular points of this system.

Let us now look at the $(n - 1)$ Equations (4) which are satisfied by the coordinates of the singular points of system (5)

$$B'_i(u_k) - UA'_i(u_k) = C_i \qquad (i = 1, \ldots, n-1) \tag{7}$$

The sum total of these equations defines a line. The coordinates of du_j, an element of this line, are determined from the equation

$$(B''_{ij} - UA''_{ij})\, du_j = 0 \qquad (i = 1, \ldots, n-1;\ j = 1, \ldots, n) \tag{8}$$

We shall assume that the equations which make up this system remain linearly independent when $U = a_\alpha$. This can always be attained by changing the numeration of the equations as a_α is a simple root of Equation (2) and therefore $n - 1$ linearly independent variables can be found from Equations (3).

Because U is close to $a_\alpha(u_k)$, the direction of an element of curve (7) in the neighborhood of point S_1 with coordinates u_k^+ is close to the direction of an element of a curve determined by Equations (3). Therefore, at some section of curve (7) in the neighborhood of point S_1 there takes place a monotonic variation $a_\alpha(u_k)$, and the derivative in a_α taken along the length of the arc of curve (7) differs from zero, i.e. the corresponding derivative taken along curve (3) differs from zero. Because U is close to $a_\alpha(u_k^+)$, on curve (7) it is possible to find a point at which $U = a_\alpha(u_k)$.

Let us look at the variation $\check{C}_n \equiv B'_n - UA'_n$ along curve (5)

$$d\check{C}_n = (B''_{nj} - UA''_{nj})\, du_j \tag{9}$$

where du_j are determined from Equations (8). Because it is possible to take as du_j minors of matrix $\| B''_{ij} - UA''_{ij} \|$ supplementary to the elements of the last row multiplied by ds, where s is some parameter along curve (7), it follows from Equation (9) that

$$\frac{d\check{C}_n}{ds} = | B''_{ij} - UA''_{ij} | \tag{10}$$

In the case of points close to the surface $a_\alpha(u_k) = U$ if we neglect higher orders of differences $U - a_\alpha$, we obtain

$$\frac{d\check{C}_n}{ds} = (U - a_\alpha) \frac{\partial}{\partial a} | B''_{ij} - aA''_{ij} |$$

Therefore, when $a_\alpha = U$

$$\frac{d^2\check{C}_n}{ds^2} = -\frac{da_\alpha}{ds} \frac{\partial}{\partial a} | B''_{ij} - aA''_{ij} | \neq 0$$

because $da_\alpha/ds \neq 0$ along curve (3), and therefore, also along curve (7) and because a_α is a simple root of Equation (2).

Thus the derivative $d\check{C}_n/ds$ changes sign and $\check{C}_n$ attains an extremum value C_n^* when, if we alter s, the value of $a_\alpha(u_k)$ goes through the value U, i.e. at the point at which curve (7) intersects the surface $a_\alpha(u_k) = U$. The value $C_n \equiv B'_i(u_k^+) - UA'_i(u_k^+)$ is sufficiently close to C_n^* (because point S_1 is close to surface $a_\alpha(u_k) = U$), therefore, on the other side of the surface $a_\alpha(u_k) = U$ there will be a point S_2 with the same value $\check{C}_n = C_n$ as at point S_1. The coordinates of this point represent values of u_k^- which satisfy the system of Equations (4). Evidently this solution is unique in the neighborhood of point S_1, i.e. in this neighborhood on curve (7) there exists only one extremum of quantity $\check{C}_n$.

Because points S_1 and S_2 lie on different sides of surface $a_\alpha(u_k) = U$, then at point S_2 we have $U < a_\alpha(u_k)$. If we fix the values U, C_1, C_2, $\ldots$, C_{n-1}, values of C_{n1} and C_{n2} will be found as close as we like to $C_n{}^*$ one of which is greater than $C_n{}^*$ and the other is less, such values that when $C_n = C_{n1}$ there exist two points S_1 and S_2 whose coordinates turn out to be solutions of Equations (4) when $C_n = C_n{}^*$ these two points run together into one point S, whilst when $C_n = C_{n2}$ system (4) has no solution in the neighborhood of point S.

We now deal with the behavior of the integral curves of system (5) in the neighborhood of the singular points. If we linearize Equations (5) in the neighborhood of one of the singular points S_r, we get

$$L_{ij}\frac{du_j}{d\xi} = P_i{}^{*\prime}(S_r) \tag{11}$$

where

$$P^*(S_r) = P''_{ij}(S_r)\,\Delta u_i\,\Delta u_j, \qquad \Delta u_k = u_k - u_k(S_r)$$

If by a linear transformation of variables Δu_k we reduce the quadratic form $P^*(S_r)$ to a sum of squares, the number m of positive coefficients of the squares, called positive indices of inertia, is equal to the number of different inequalities $a_\beta(S_r) < U$.

In actual fact if when U changes the indices of inertia of the quadratic form $P''_{ij}(S_r)\,\Delta u_i\,\Delta u_j$ vary, then simultaneously the determinant $|\,P''_{ij}(S_r)\,| = |UA''_{ij}(S_r) - B''_{ij}(S_r)\,|$ should vanish. If $U = a_\beta(S_r)$, where a_β is a simple root of equation $|B''_{ij}(S_r) - aA''_{ij}(S_r)| = 0$ then the rank of the matrices $\|\,B''_{ij}(S_r) - a_\beta A''_{ij}(S_r)\,\|$ equals $n - 1$. Therefore when we represent $P^*(S_r)$ as a sum of squares only one of the coefficients vanishes. Because the matrix $\|\,A''_{ij}\,\|$ is positive definite when $U = \infty$ the positive index of inertia equals n, and when $U = -\infty$ it is zero. Therefore when we represent $P^*(S_r)$ as a sum of squares one of the positive coefficients is replaced by a negative one, when U, on decreasing, goes through the simple root $a_\beta(S_r)$. Multiple roots may be regarded as simple ones which run into each other.

We will demonstrate that if the positive index of inertia of the non-decaying quadratic form $P^* = m$, the integral curves which emanate from point S_r make up a surface of m dimensions*.

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* This statement follows from [3]; with the type of system of equations with which we are dealing, the proof is considerably simplified.

To do this we construct on the surface $P(u_k) = P(S_r) + \varepsilon(\varepsilon > 0)$ which in the neighborhood of point S_r can be represented by the equation $P^*(S_r) = \varepsilon$, the closed $(m-1)$-dimensional surface Σ_1. The integral curves which emanate from points on the surface Σ_1 in the direction of increasing P create an m-dimensional surface Σ_2. None of these integral curves can enter point S_r, because for points on the surface Σ_1 the inequalities $P(u_k) > P(S_r)$ are fulfilled whilst along the integral curves further increase in P takes place. Now suppose $\varepsilon \to 0$ and all points on the surface Σ_1 tend to coincidence with point S_r. The surface Σ_2 then tends to some limiting surface Σ_3 which is actually the m-dimensional surface consisting of integral curves which emanate from point S_r. In a similar manner it can be shown that there exists an $(n-m)$-dimensional surface consisting of integral curves which enter this point.

Now let us look at the surface $P(u_k) = G$, $G = \text{const}$, in the space u_k. The function P depends on the parameters $U, C_1, C_2, \ldots, C_n$. Because everywhere in what follows all these parameters except C_n will be regarded as constant, to emphasise the dependence of the given surface on C_n we will in some cases write down its equation in the form $P(u_k, C_n) = G$.

We will say that two regions have one and the same topological type (homeomorphic), if one can be transformed into the other by a continuous transformation whose sign does not change.

Evidently the topological type of space or region $P(u_k, C_n) \geqslant G$ will not vary with change in G if here the surface $P(u_k, C_n) = G$ does not go through any stationary points of the function $P(u_k, C_n)$. In actual fact with infinitely small change in G it is possible to construct a transformation of the surface $P(u_k, C_n) = G$ in the neighborhood $P(u_k, C_n) \geqslant G$, which is single signed and continuous if grad $P(u_k, C_n)$ does not vanish on the surface $P(u_k, C_n) = G$. In the same manner if the surface $P(u_k, C_n) = G$ does not go through any stationary points of the function $P(u_k, C_n)$ then with a sufficiently slow change in C_n, the topological type of region $P(u_k, C_n) \geqslant G$ will not change.

If G, on increasing, goes through stationary values of $P(S_r)$ corresponding to point S_r with a nondecaying quadratic form $P^*(S_r)$, then as is known from [9] the Betti number of $(m-1)$ dimensions increases by one** or the m-dimensional Betti number of region $P(u_k) \geqslant G$ decreases by

** The Betti number and the homology are regarded everywhere in this work only in terms of mod 2, and this will not be repeated each time in what follows for the sake of brevity. The l-dimensional Betti number of any space or region will be called the maximum number of homologously independent l-dimensional cycles which can be constructed within this region. Cycles refer to closed surfaces (surfaces having one, where m is a positive index of inertia of the quadratic form $P^*(S_r)$. In the first case point S_r is called a point of the growing or increasing type; in the second, it is called a point of the decaying type.

We will consider that when G, upon changing, does not go through stationary values of the function $P(u_k)$, cycles which constitute a wholly homologously independent system, vary within region $P(u_k) \geqslant G$ when G changes continuously (in an arbitrary manner). How this continuous transformation takes place is irrelevant as far as we are concerned for if G does not go through stationary values, all cycles attained by continuous deformation from any one are homologously identical. Cycles which with $G = P(S_r)$ do not go through point S_r may be regarded as varying in a continuous manner when G goes through the values $P(S_r)$. They will then remain homologously independent, i.e. region $P(u_k) \geqslant G$ decreases when G is increased.

The full system of homologously independent cycles of region $P(u_k) \geqslant G$ can always be so chosen that for values of G close to $P(S_r)$, all cycles with the exception possibly of one, lie outside a fairly small region surrounding point S_r. This cycle is additional to the full system of cycles, when the point S_r is a point of the increasing type or it decays from this system if point S_r is a point of the decaying type.

If the cycle of the greatest number of dimensions $m - 1$ lying on surface $P^*(S_r) = \varepsilon$ which represents the surface $P(u_k) = G$, in the neighborhood of point S_r, is not homologously zero in region $P(u_k) \geqslant G$, it is not homologous with the other cycles which constitute a total or fully homologously independent system, and it may be taken as a cycle which is supplementary to this system when G goes through value $P(S_r)$.

If this $(m - 1)$-dimensional cycle is homologously zero in region $P(u_k) \geqslant G$ the m-dimensional surface, which contains this cycle as a

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no boundaries). The sum total of l-dimensional cycles which lie in any region is called "homologously independent" if in this region it is not possible to construct an $(l + 1)$-dimensional space, whose boundary consists of these cycles (not necessarily all of them). An l-dimensional cycle is called homologously zero if there exists an $(l + 1)$-dimensional surface whose boundary it appears to be. For further details refer to [10].

We will term a system of homologous mutually independent cycles lying in a given region "full" if any cycle not belonging to this system is homologously dependent in the given region on cycles which constitute this system.

boundary and lies in region $P(u_k) \geqslant G$, when $G < P(S_r)$ can be completed up to an m-dimensional cycle, by adding to it a surface which describes in space an $(m - 1)$-dimensional cycle, when with decreasing G it tends towards point S_r. An m-dimensional cycle constructed in this manner is not homologously zero in region $P(u_k) \geqslant G$ when $G < P(S_r)$.

If it turns out that there exist several m-dimensional cycles which go through the neighborhood of point S_r and they are not homologous with respect to each other it is always possible to choose a system of cycles which is homologously equivalent to these cycles and which contains only one m-dimensional cycle which passes through the neighborhood of point S_r (one can take as this cycle any of the m-dimensional cycles which go through the neighborhood of point S_r). We will therefore consider that the total system of independent cycles in region $P(u_k) \geqslant G$ when $G < P(S_r)$ contains only one m-dimensional cycle which passes through the region close to point S_r, and which indeed represents a cycle which emanates from the full system of homologously independent cycles, when G goes through value $P(S_r)$ (for brevity we will say that it breaks away at point S_r).

Notice that if any m-dimensional cycle breaks away at point S_r any cycle which is homologous thereto will break away at this point and conversely if any m-dimensional cycle does not break away at point S_r then this is true for all cycles which are homologous thereto. This follows from the fact that an m-dimensional cycle breaking away at point S when $G = P(S_r)$ cannot be homologously dependent on cycles which go through this point, i.e. no surfaces exist whose dimensions exceed m and which lie in region $P(u_k) \geqslant P(S_r)$ and pass through point S_r.

Let us look at regions *1* and *2*, determined respectively by inequalities $P(u_k, C_{n1}) \geqslant G$ and $P(u_k, C_{n2}) \geqslant G$, where the values of C_{n1} and C_{n2} which were deduced before, are sufficiently close to C_n^*.

Then if G varies over an interval of values containing $G^* = P(S, C_n^*)$, the surface $P(u_k, C_{n1}) = G$ goes through two stationary points S_1 and S_2 which lie in the neighborhood of point S, whilst the surface $P(u_k, C_{n2}) = G$ when G varies in the same interval, does not touch the stationary points in the neighborhood of point S.

Let us assume that for C_n enclosed between C_{n1} and C_{n2}, the surface $P(u_k, C_n) = G$ does not go through any other stationary points, that is apart from S_1 and S_2, when G varies over the given range of values.

Then a change in the topological type of region $P(u_k, C_n) \geqslant G$ can take place only when the surface $P(u_k, C_n) = G$ goes through points S_1 and S_2.

If the above assumption is not fulfilled it is always possible to regard the finite neighborhood of point S given by inequality $F(u_k) \geqslant 0$, as not containing other stationary points apart from S_1 and S_2, and in all the discussions that follow it is possible to watch for changes in the topological type of region $P(u_k,\ C_n) \geqslant G,\ F(u_k) \geqslant 0$.

Therefore, if $G \neq G^*$ and C_{n1} is sufficiently close to C_{n2} regions *1* and *2* then have exactly the same topological type. If we fix C_{n1} and C_{n2} and vary G, the topological type of region *1* will change as G goes through the stationary values $P(S_1,\ C_{n1})$ and $P(S_2,\ C_{n1})$ whilst the topological type of region *2* will not change. Therefore the regions under discussion will have the same topological type for $G < P(S_i,\ C_{n1})$ and for $G > P(S_f,\ C_{n1})$, where $i,\ f = 1,\ 2$, so that $P(S_i,\ C_{n1}) \leqslant P(S_f,\ C_{n1}) \leqslant$ It follows from this that changes in Betti numbers of region *1* when $G = P(S_1,\ C_{n1})$ and $G = P(S_2,\ C_{n2})$ should compensate each other.

We will demonstrate that point S_f is a point of the decaying type. If this were not so, when G went through the value $P(S_f,\ C_{n1})$ the totality of homologously independent cycles of region *1* would be supplemented by the new cycle R_1, and this cycle can be so chosen that for values of G sufficiently close to $P(S_f,\ C_{n1})$, it lies within an arbitrarily small neighborhood of point S_f.

Because when $G > P(S_f,\ C_{n1})$ regions *1* and *2* possess the same topological type, in region *2* of the cycle constructed above another corresponding cycle R_2 can be placed which is not homologously zero within this region.

If the values of C_{n1} and C_{n2} are sufficiently close, the cycle R_2 can be so chosen that it will be close to R_1 and therefore when values of G are close to $P(S_f,\ C_{n1})$ it will be in a small neighborhood D of point S_f.

With decreasing G region *2* will expand; and since in the region with which we are dealing $|\text{grad}\ P(u_k,\ C_{n1})|$ is bounded, the velocity is not zero and has a lower bound. Therefore with only a small decrease in G the whole neighborhood D will belong to region *2* and the cycle R_2 will become homologous to zero. This however is not possible because the topological type of region *2* cannot alter.

It follows that at points S_i and S_f the function $P(u_k,\ C_{n1})$ takes on different values, since if this were not so either of them might be taken as S_f, and therefore both must be points of the decaying type, so that when G passes through this doubly critical value the topological type of region *2* should change. Because the change in Betti numbers of region *1*, when G goes through stationary values, should be reversible, it follows that point S_i is a growing type of point.

Suppose that at point S_i the positive index of the quadratic forms $P^*(S_i, C_{n1})$ equals m. When G goes through the value $P(S_i, C_{n1})$ the whole system of region *1* cycles is supplemented by one $(m - 1)$-dimensional cycle. When G goes through the value $P(S_f, C_{n1})$ from the whole gamut of homologous independent region *1* cycles the $(m - 1)$-dimensional cycle should vanish. It follows from this that the positive index of inertia of the quadratic form $P^*(S_f, C_{n1})$ equals $m - 1$.

If we know the positive indices of inertia of the quadratic forms $P^*(S_i, C_{n1})$ and $P^*(S_f, C_{n1})$ it is possible to conclude that $U > a_m(S_i)$ and $U < a_m(S_f)$, i.e. that $i = 1$, $f = 2$, $m = \alpha$ and $P(S_1, C_{n1}) < P(S_2, C_{n2})$.

We will show that an $(\alpha - 1)$-dimensional cycle arising when $G = P(S_1, C_{n1})$ in the neighborhood of point S_1, goes through point S_2 when $G = P(S_2, C_{n1})$, whilst with further increase of G it breaks away**. If this cycle did not break up at point S_2 then when $G > P(S_2, C_{n1}$ in region *1* there would exist a cycle not homologous to zero which is placed or located in some region which contains points S_1 and S_2, the dimensions of which tend to zero when $C_{n1} \to C_n^*$, $G \to G^*$ on condition that $G > P(S_2, C_{n1})$.

Actually as an example of a cycle arising or generated at point S_1 one can take the cycle lying on the surface $P(u_k) = G$. If when G changes this cycle deforms along the vector grad $P(u_k)$ and remains on surface $P(u_k) = G$, this deformation will be continuous (because in accordance with the assumption the cycle cannot go through point S_2 and it will not extend beyond the limits of a fairly small region which contains points S_1 and S_2. Because when $G > P(S_2, C_{n1})$ regions *1* and *2* are of the same topological type and the boundaries of these regions are as close as we like to each other, if C_{n1} and C_{n2} are sufficiently close to each other then in region *2* there should also exist a cycle not homologous to zero, in some small region containing points S_1 and S_2. However this is impossible because in this case with only small decreases in G the topological type of region *2* would change.

It now follows that an $(\alpha - 1)$-dimensional cycle, arising for $G = P(S_1, C_{n1})$ in the neighborhood of point S_1 with $G = P(S_2, C_{n1})$, goes through point S_2. This situation does not depend on how the given cycle has deformed when G has changed from $P(S_1, C_{n1})$ to $P(S_2, C_{n2})$, for if any cycle breaks up, then all the cycles break up which are homologous thereto.

** In the three-dimensional case ($n = 3$) this statement becomes self evident from the geometry of the surface $P(u_k) = G$ (see [7].

A cycle $\Sigma(G)$ can be chosen as an $(\alpha - 1)$-dimensional cycle which is not homologous to zero in region *1* and arising for $G = P(S_1, C_{n1})$, at point S_1, and it is the intersection of the surfaces made up of the integral curves emanating from point S_1 with the surface $P(u_k, C_{n1}) = G$. The deformation of this cycle is given by Equations (5) and is continuous over all finite points of space which are not stationary points of the function $P(u_k, C_{n1})$.

On condition that $|L_{ij}| \neq 0$ the integral curves of Equation (9) in any bounded, closed region make a finite angle with a surface $P(u_k, C_{n1}) = G$. Therefore if we choose the difference $P(S_2, C_{n1}) - P(S_1, C_{n1})$ to be sufficiently small (and this is allowed for by the closeness between C_{n1} and C_{n2}) it is possible to show that the integral curves emanating from point S_1 as G varies within the limits $P(S_1, C_{n1}) < G < P(S_2, C_{n1})$ will not go outside the previously given region.

Besides, when G is varied over the given interval the surface $P(u_k, C_{n1}) = G$ does not go through the stationary points of the function $P(u_k, C_{n1})$. Therefore according to what has been said the cycle $\Sigma(G)$ for $G = P(S_2, C_{n1})$ goes through point S_2.

It follows that there exists at least one integral curve which connects points S_1 and S_2. This integral curve is the continuous solution which reduces to u_k^+ when $x = \infty$ and to u_k^- when $x = -\infty$.

The above proof is still valid for shock waves of finite amplitude if the shock wave corresponds to a transformation $S_1 \to S_2$ such that for given $U, C_1, C_2, \ldots, C_n$ between $P(S_1)$ and $P(S_2)$ there are no other stationary values of the function $P(u_k)$ and the cycle $\Sigma(G)$ cannot break up at an infinitely distant point when G varies between the limits $P(S_1) < G < P(S_2)$. The latter condition may be fulfilled either through the properties of dissipation coefficients L_{ij} or by the surface $P(u_k) = G$ for $P(S_1) < G < P(S_2)$ not containing the point at infinity.

Suppose now that $L_{ij} \to 0$.** If the cosine of the angle between the vectors $\{z_i\}$ and $\{L_{ij}z_i\}$ does not tend to zero for any values of z_i, then the cosine of the angle between vectors $\{\Lambda_{ij}y_i\}$ and $\{y_j\}$ will not tend to zero for any value of y_i, where Λ_{ij} denotes elements of the matrices which are inverse to $\|L_{ij}\|$. To prove this it is sufficient to put $y = L_{ij}z_j$.

** In the case of magnetohydrodynamics with dissipation given by diagonal matrix L_{ij} such a limiting transformation in the solution representing the structure of a shock wave is dealt with in [11].

Equations (5) and (6) can be written down thus

$$\frac{du_i}{d\xi} = \Lambda_{ij}P'_j, \qquad \frac{dP}{d\xi} = \Lambda_{ij}P'_iP'_j \tag{12}$$

It follows that the integral curves for $L_{ij} \to 0$ will make a finite angle with the surfaces $P(u_k) = G$. Therefore on the integral curve u_i they will be continuous functions of P, and when P varies between the limits $P(S_1) < P < P(S_2)$ they will not go out of a given region Q which contains points S_1 and S_2. Besides it is possible to isolate a δ-region close to singular points S_1 and S_2 such that for P differing from $P(S_1)$ and $P(S_2)$ by less than ε, the corresponding point on the integral curve connecting S_1 and S_2 lies in one of these δ-regions. Let us look at a section of the integral curve corresponding to changing P over the range $P(S_1) + \varepsilon \leqslant P \leqslant P(S_2) - \varepsilon$. As this interval does not contain stationary values of the function P, in region Q from which the δ-regions of singular points have been isolated, the following inequality is fulfilled

$$\sqrt{\sum_i P'^2_i} > \varepsilon_1$$

If all the $L_{ij} \to 0$, then for any fixed vector $\{z_i\}$ the modulus of vector $\{\Lambda_{ij}z_j\}$ tends to infinity. This follows from the fact that for any fixed vector $\{y_i\}$ the modulus of vector $\{L_{ij}y_j\}$ tends to zero. As in this case the cosine of the angle between vectors $\{\Lambda_{ij}z_j\}$ and $\{z_i\}$ does not tend to zero for any value of $\{z_i\}$ we have

$$\frac{dP}{d\xi} = \Lambda_{ij}P'_iP'_j \to \infty$$

and variations ξ over the given section of the integral curve tend to zero

$$\xi_2 - \xi_1 = \int\limits_{P(S_1)+\varepsilon}^{P(S_2)-\varepsilon} \frac{dP}{\Lambda_{ij}P'_iP'_j} \to 0$$

Thus the length of a section of minimum length which contains all the points in which at least one value of u_i differs from u_i^+ and u_i^- by an amount greater than δ, tends to zero when $L_{ij} \to 0$.

If together with L_{ij} the cosine of the angle between vectors $\{\Lambda_{ij}P'_j\}$ and $\{P'_i\}$ tends to zero, the expression $\Lambda_{ij}P'_iP'_j$ may fail to tend to infinity and the difference $\xi_2 - \xi_1$ may remain finite or it may tend to infinity. In this case the modulus of the vector $\{\Lambda_{ij}z_j\}$ for any fixed value of $\{z_i\}$ tends to infinity; from the first Equation (12) it follows that at each point of section $[\xi_1\xi_2]$

$$\sqrt{\sum_i \left(\frac{du_i}{d\xi}\right)^2} \to \infty, \qquad \int_{\xi_1}^{\xi_2} \sqrt{\sum_i \left(\frac{du_i}{d\xi}\right)^2}\, d\xi \to \infty$$

if $\xi_2 - \xi_1$ does not tend to zero when $L_{ij} \to 0$. The solution may possess for instance a quasi-periodic character over the section $[\xi_1, \xi_2]$, whilst the period tends to zero when $L_{ij} \to 0$. An example of such a limiting transformation can easily be constructed for the case when the matrix $\| L_{ij} \|$ is such that $L_{ij} = -L_{ji}$ and when $i \neq j$ [12], where the limiting transformation $L_{ij} \to 0$ is regarded in magnetohydrodynamics as a generalized Ohm's law, or when the matrix $\| L_{ij} \|$ is orthogonal. However, if the matrix $\| L_{ij} \|$ is symmetrical then whenever any of its elements tend to zero the difference $\xi_2 - \xi_1$ tends to zero because in this case $\Lambda_{ij} z_i z_j \to \infty$ for any choice of z_i.

In some cases the reduction of a given concrete system of partial differential equations to form (1) may be rather more difficult than obtaining ordinary equations in form (5), which are satisfied by the solutions which depend on $\xi = Ut - x$ (in the latter case it is convenient to choose as P the flux of entropy – see for instance [8] where equations which describe steady magnetohydrodynamic flows in form (5) are derived. Therefore for convenience in presentation we will formulate the result obtained for the system of ordinary differential equations of form (5).

Let the function $P(u_k)$ depend continuously on a parameter q and suppose that for $q < q^*$ function $P(u_k)$ has stationary points S_1 and S_2, in the neighborhood of which the function $P(u_k)$ is represented by non-decaying quadratic forms relative to the deviation of the variables from their values at points S_1 and S_2.

Let the points S_1 and S_2 tend to coincidence when $q \to q^*$, and this takes place at some point S, whilst it is possible to separate a region round point S such that for values of q close to q^* there are, in this region, no other stationary points of function $P(u_k)$ except S_1 and S_2. Suppose when $q > q^*$ in this region there are no stationary points of function $P(u_k)$. Then if q is close to q^*, whilst $q < q^*$, there will exist at least one integral curve which connects the singular points S_1 and S_2 of system (5).

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